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LETTER TO THE EDITOR

On the partition function of the Ising model in a magnetic field on an arbitrary lattice

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Abstract. We show that the partition function of the Ising model in a magnetic field on an arbitrary lattice in arbitrary dimensions is expressed by an ensemble average of the partition functions of random-bond Ising models without field.

It is well known that the Ising model in a magnetic field in dimensions larger than 1, as well as the zero field model in dimensions $d \geqslant 3$, has not been solved so far. Here we present exact relations between the partition function of the Ising model in a field and the partition functions of random-bond Ising models in zero field in an arbitrary lattice (defined below). We shall use the polygon picture in the low-temperature expansion of the former. Then, to go further, we will see that it is necessary to measure the volume of a polygon in some way. Introducing auxiliary random (Ising) variables on bonds of the original lattice and statistical weights for these variables, we will be able to do this. The final results are equations (7) and (10).

Consider a general lattice (or graph) $\mathcal{L}$ in d-dimensions ($d = 1, 2, \ldots$); a lattice $\mathcal{L}$ here means a collection of (0-dimensional) sites and (1-dimensional) bonds; each bond defines a connection of two sites such that the lattice is embedded into a d-dimensional space *consistently*. (The definition here may be loose from a mathematical point of view). The square lattice, the triangular lattice, the cubic lattice, and so on, are special examples of the lattice thus defined. Assume a cyclic boundary condition. Denote the set of all sites V, the set of all bonds B. Set $N = \sharp V$ and $M = \sharp B$.

On each site $\alpha \in V$ put an Ising variable $\sigma_\alpha \in \{1, -1\}$. We consider the Ising model on this lattice $\mathcal{L}$ with a site-dependent magnetic field:

$$Z_N(H) = \sum_{\sigma \in C} \exp\left(\sum_{(\alpha,\beta)\in B} K_{\alpha\beta}\sigma_\alpha\sigma_\beta + \sum_{\alpha \in V} H_\alpha \sigma_a \right). \tag{1}$$

Here the first summation is over all configurations of Ising variables (2^N terms in all), $K_{\alpha\beta}$ the interaction strength attached to the bond (α, β), H_α the magnetic-field strength attached to the site α, and $H = (H_1, H_2, \ldots, H_N)$. We allow the $K_{\alpha\beta}$ and H_α to take

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complex values. The partition function (1) has a global symmetry $Z_N(H) = Z_N(-H)$ where $-H = (-H_1, -H_2, \ldots, -H_N)$. The *low-temperature expansion* of $Z_N(-H)$ (cf e.g. [1], chapter 6) yields

$$
\begin{aligned}
Z_N(H) &= \prod_{(\alpha,\beta)} \mathrm{e}^{K_{\alpha\beta}} \prod_{\alpha} \mathrm{e}^{H_\alpha} \sum_{\sigma\in C} \prod_{(\alpha,\beta)} x_{\alpha\beta}^{\frac{1-\sigma_\alpha\sigma_\beta}{2}} \prod_{\alpha} y_\alpha^{\frac{1+\sigma_\alpha}{2}} \\
&= \prod_{(\alpha,\beta)} \mathrm{e}^{K_{\alpha\beta}} \prod_{\alpha} \mathrm{e}^{H_\alpha} \sum_{\sigma\in C^+} \prod_{(\alpha,\beta)} x_{\alpha\beta}^{\frac{1-\sigma_\alpha\sigma_\beta}{2}} \left(\prod_{\alpha} y_\alpha^{\frac{1+\sigma_\alpha}{2}} + \prod_{\alpha} y_\alpha^{\frac{1-\sigma_\alpha}{2}} \right)
\end{aligned}
$$

where $x_{\alpha\beta} = \mathrm{e}^{-2K_{\alpha\beta}}$, $y_\alpha = \mathrm{e}^{-2H_\alpha}$. We have decomposed the set of configurations as $C = C^+ \sqcup (-C^+)$ (disjoint union). For each bond (α, β) of $\mathcal{L}$ there is a $(d-1)$-dimensional hyper-face (a $(d-1)$-face, for short), denoted (α, β), transverse to the bond. Let $\mathcal{L}^*$ be a dual lattice of $\mathcal{L}$ defined by the collection of all such $(d-1)$-faces. A d-dimensional hyper-cubic element (a d-cube, for short) of $\mathcal{L}^*$ is numbered uniquely by the site index α of $\mathcal{L}$. A *polygon* P in $\mathcal{L}^*$ is, by definition, a set of $(d-1)$-faces such that every $(d-2)$-dimensional hyper-edge ($(d-2)$-edge, for short) is shared by even numbers (i.e. $0, 2, 4, \ldots$) of $(d-1)$-faces of P. Note that a polygon is not necessarily connected. We shall consider only polygons which have definite interior regions; i.e. we neglect polygons that go across the boundary an odd number of times. Let $\mathcal{P}$ be the set of all such polygons in $\mathcal{L}^*$. For a polygon $P \in \mathcal{P}$, let the same symbol P be the set of all $(d-1)$-faces of P, $\mathrm{In}P$ ($\overline{\mathrm{In}P}$) the set of all d-cubes interior (exterior) of P. Then, we can write the partition function in terms of polygons:

$$
Z_N(H) = \prod_{(\alpha,\beta)} \mathrm{e}^{K_{\alpha\beta}} \prod_{\alpha} \mathrm{e}^{H_\alpha} \sum_{P\in\mathcal{P}} \prod_{(\alpha,\beta)\in P} x_{\alpha\beta} \left(\prod_{\alpha\in\mathrm{In}P} y_\alpha + \prod_{\alpha\in\overline{\mathrm{In}P}} y_\alpha \right). \tag{2}
$$

The definition of the interior of P is not relevant in this expression. The summation in (2) can be interpreted as a generating function of the numbers of polygons having $(d-1)$-faces with weights $x_{\alpha\beta}$ and interior d-cubes with weights y_α (the first term in parentheses). The presence of such factors counting the *volume* of the interior of P (the volume of P, for short) complicates calculations of the partition function; here, we think of $\log y_\alpha$ as the volume of a d-cube α. For instance, when $\mathcal{L}$ is a two-dimensional square lattice and *a magnetic field is absent*, we can convert (2) into a Pfaffian (even for arbitrary $x_{\alpha\beta}$'s; this is a simple corollary of the work in [2]); but, if $H_\alpha \neq 0$ we cannot do that.

In order to measure the volume of a polygon, we prepare an auxiliary Ising model. On each $(d-1)$-face (α, β) put an Ising variable $\tau_{\alpha\beta} \in \{1, -1\}$. Consider an Ising model defined by

$$
I_M = \sum_{\tau} \exp\left(\sum_{\alpha} \beta_\alpha \underbrace{\tau \cdots\cdots \tau}_{\text{round a } d \text{ cube } \alpha} \right). \tag{3}
$$

The summation in the exponential is over all d-cubes in $\mathcal{L}^*$; the Ising variables round each d-cube α interact with strength β_α. It is easy to show that the product of all variables on $(d-1)$-faces of a polygon P has a statistical average

$$
\left\langle \prod_{(\alpha,\beta)\in P} \tau_{\alpha\beta} \right\rangle_\beta = \frac{\prod_{\alpha\in\mathrm{In}P} y_\alpha + \prod_{\alpha\in\overline{\mathrm{In}P}} y_\alpha}{1 + \prod_\alpha y_\alpha} \tag{4}
$$

where $y_\alpha = \tanh\beta_\alpha$ (to show this, use the *high-temperature expansion*; cf [1], chapter 6). In this way we can measure the volume of P using a special Ising model (3).

Now let us combine the results (4) and (2) to find an alternative expression of $Z_N(H)$. Set $H_\alpha = 0$ in the original model (2); but replace each bond strength $x_{\alpha\beta}$ with $x_{\alpha\beta}(\tau) = \tau_{\alpha\beta}x_{\alpha\beta}$, which depends on a random variable $\tau_{\alpha\beta} \in \{1, -1\}$:

$$\widetilde{Z}_N[\tau] = 2\prod_{(\alpha,\beta)} \mathrm{e}^{K_{\alpha\beta}} \sum_{P\in\mathcal{P}} \prod_{(\alpha,\beta)\in P} x_{\alpha\beta}(\tau). \tag{5}$$

Each term in this summation has exactly the same form as the product in the correlator (4). Thus, averaging both sides of (5) with weights given in (3), we get

$$\left\langle \widetilde{Z}_N[\tau] \right\rangle_\beta = \frac{2}{1+\prod_\alpha y_\alpha} \prod_{(\alpha,\beta)} \mathrm{e}^{K_{\alpha\beta}} \sum_{P\in\mathcal{P}} \prod_{(\alpha,\beta)\in P} x_{\alpha\beta} \left(\prod_{\alpha\in \mathrm{In}P} y_\alpha + \prod_{\alpha\in \overline{\mathrm{In}P}} y_\alpha \right). \tag{6}$$

Comparing with (2), we find an interesting identity

$$Z_N(H) = \cosh\left(\sum_\alpha H_\alpha\right) \cdot \left\langle \widetilde{Z}_N[\tau] \right\rangle_\beta \tag{7}$$

where $\tanh\beta_\alpha = \mathrm{e}^{-2H_\alpha}$ $(= y_\alpha)$; the LHS is the partition function of the original Ising model in a field; the RHS is a statistical average over an ensemble of the partition functions of random-bond Ising models without field; the magnetic field in the original model (1) is absorbed in the statistical weights in (3).

To be precise, equation (5) differs from the partition function of the random-bond Ising model in normalization. The precise equation is

$$Z_N[\tau] = \sum_{\sigma\in C} \exp\left(\sum_{(\alpha,\beta)\in B} K_{\alpha\beta}(\tau)\sigma_\alpha\sigma_\beta \right) = 2\prod_{(\alpha,\beta)} \mathrm{e}^{K_{\alpha\beta}(\tau)} \sum_{P\in\mathcal{P}} \prod_{(\alpha,\beta)\in P} x_{\alpha\beta}(\tau) \tag{8}$$

where

$$K_{\alpha\beta}(\tau) = \begin{cases} K_{\alpha\beta} & (\tau_{\alpha\beta} = +1) \\ K_{\alpha\beta} - \mathrm{i}\pi/2 & (\tau_{\alpha\beta} = -1). \end{cases}$$

This is an unphysical model in the sense that $K_{\alpha\beta}(\tau)$ can have an imaginary part.

It is possible to avoid the use of unphysical interactions. To do so, let us consider the *dual* of (8):

$$Z^*_M[\tau] = \sum_{\sigma^*} \exp\left(\sum_{(\alpha,\beta)} L_{\alpha\beta}(\tau) \underbrace{\sigma^* \cdots\cdots \sigma^*}_{\text{round a } (d-1) \text{ face}} \right). \tag{9}$$

The Ising variables σ^*_i live on $(d-2)$-edges i of $\mathcal{L}^*$. Those round each $(d-1)$-face interact. The high-temperature expansion yields

$$\begin{aligned} Z^*_M[\tau] &= \prod_{(\alpha,\beta)} \cosh L_{\alpha\beta}(\tau) \sum_{\sigma^*} \prod_{(\alpha,\beta)} (1 + z_{\alpha\beta}(\tau)\sigma^* \cdots \sigma^*) \\ &= 2^M \prod_{(\alpha,\beta)} \cosh L_{\alpha\beta}(\tau) \sum_{P\in\mathcal{P}} \prod_{(\alpha,\beta)\in P} z_{\alpha\beta}(\tau) \end{aligned}$$

where $z_{\alpha\beta}(\tau) = \tanh L_{\alpha\beta}(\tau)$. This gives the same polygon-expansion as in (8) if we set $L_{\alpha\beta}(\tau) = \tau_{\alpha\beta} L_{\alpha\beta}$, $\mathrm{e}^{2L_{\alpha\beta}} = 1/\tanh K_{\alpha\beta}$. Comparing with (5), we have

$$Z_M^*[\tau] = 2^{M-1} \prod_{(\alpha,\beta)} \mathrm{e}^{-K_{\alpha\beta}} \cosh L_{\alpha\beta} \cdot \widetilde{Z}_N[\tau]$$

and, therefore,

$$Z_N(H) = \frac{1}{2^{M-1}} \prod_{(\alpha,\beta)} \frac{\mathrm{e}^{K_{\alpha\beta}}}{\cosh L_{\alpha\beta}} \cdot \cosh\Bigl(\sum_\alpha H_\alpha\Bigr) \cdot \Bigl\langle Z_M^*[\tau] \Bigr\rangle_\beta . \tag{10}$$

This is the alternative to equation (7).

Let us summarize and comment on the result. We found exact relations between the Ising model in a field and random-bond Ising models in zero field: equations (7) and (10); the randomness was introduced in bonds of the original model, and the magnetic field was absorbed in the statistical weights of the random variables. If, for example, we substitute the transfer-matrix expressions of the partition functions into the RHS of (7) or (10) we obtain an expression of $Z_N(H)$ in terms of transfer matrices which contain random variables. Although the relations might be useless to solve the model exactly, recognizing such relations would make our understanding of the problem deeper. Finally, we remark that the use of the Ising model (3) is inspired by the work by Wegner [3] (for a review, see section V in [4]); Wegner considered an Ising model with variables on bonds of the d-dimensional (regular) lattice, Boltzmann weights given by an interaction-round-a-face, $\mathrm{e}^{\sum \beta\sigma\sigma\sigma\sigma}$, and found asymptotically exact expressions of the minimum area and the perimeter of a closed loop in terms of Ising-model correlation functions. We note that we can measure various characteristics of a polygon using suitable Boltzmann weights: for instance, replacing the weight in (3) with $\mathrm{e}^{\sum \gamma_{\alpha\beta}\tau_{\alpha\beta}}$ yields the *area* of hyper-surface of a polygon.

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